Feynman's Integration Trick

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

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Disclaimers

Richard Feynman didn’t invent this trick,

He popularised it by mentioning it in his autobiography

Surely you’re joking. Mr Feynman

as an entry in his box of tools. He got it from
Advanced Calculus by Fredrick S Woods
A book from 1926 which is still valued.

And I adapted this talk mainly from *Differentiation under the Integral Sign* by Leo Goldmakher

An Introduction

Starting from

$$F(t) = \int_0^\infty e^{-tx} dx$$

Differentiating by *t* we get

$$F'(t) = \int_0^\infty x e^{-tx} dx = \frac{1}{t^2}$$

Differentiating *n* times

$$\int_0^\infty x^n e^{-tx} dx = \frac{n!}{t^{n+1}}$$

The Trick

Which with *t*=1 gives Euler's integral formula for the Gamma function

$$\int_0^\infty x^n e^{-x} dx = n!$$

Something we wouldn't get just starting with e^{-x}

Feynman's trick is to introduce a second variable under the integration sign, then differentiate under the integral with respect to that variable. With a suitable choice this can often simplify the solution of the original integral.

A Simple Example

We'll evaluate

$$\int_0^1 \frac{x^2 - 1}{\log x} dx$$

You won't find anything like this in a table of standard integrals.
How on earth can such an integral ever be evaluated!

How can that log *x* be got rid of by differentiating?

Big clue - Replacing the 2 with *t* can do the job!

A Simple Example - Solution

Replacing the 2 with *t* we get
So the value of the original integral is G(2)

$$G(t) = \int_0^1 \frac{x^t - 1}{\log x} dx$$

Differentiating with respect to *t* we get

$$G'(t) = \int_0^1 \frac{x^t \log x}{\log x} dx$$

$$= \int_0^1 x^t dx = \left. \frac{x^{t+1}}{t+1} \right|_0^1 = \frac{1}{t+1} \qquad t \geq 0$$

Simple Example - finale

We want the value of $G(2)$ given that

$G(0) = 0$ and $G'(t) = \dfrac{1}{t+1}$

$$G(2) = \int_0^1 \frac{x^2 - 1}{\log x} dx = \int_0^2 \frac{dt}{t+1} = \log 3$$

A Little more Complex

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

We can see the negative half is just the same as the positive so we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_{0}^{\infty} \frac{\sin x}{x} dx$$

But how to get rid of the *x* underneath?

Solution

Introduce a factor which is one at $t = 0$
and differentiates to cancel the x

$$e^{-tx}$$

$$H(t) = \int_0^\infty \frac{\sin x}{x} e^{-tx} dx \qquad t \geq 0$$

$$H'(t) = -\int_0^\infty \sin x \; e^{-tx} dx \qquad t > 0$$

Solution continued

Using this we get easy to integrate exponentials

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

After simplifying this gives

$$H'(t) = \frac{-1}{1+t^2}$$

And a standard table of integrals gives this result for *H*

$$H(t) = C - \tan^{-1} t$$

Solved

$$\lim_{t\to\infty} H(t) = 0,\ \tan^{-1} t = \frac{\pi}{2}$$

$$\implies H(t) = \frac{\pi}{2} - \tan^{-1} t \qquad\qquad t > 0$$

Letting *t* tend to 0 from the right gives the limit $H(0) = \frac{\pi}{2}$

So we get $$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$